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Asian Resonance

Properties of Weak and Strict Stationary Time Series

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Abstract

In this paper, a time series $\{X_t(\omega),\ t \in T\}$ on (Ω , C, P) is explained. Where X is a random variable (r. v.). The properties of weak and strict stationary time series with supporting real life examples have been taken and conclusion have been drawn by testing methodology of hypothesis. Number of rainy days and rainfall data for 26 years from Aurangabad District of Maharastra State were analyzed.

Keywords : Time Series, Regression Model, Auto-Covariance, Auto-Correlation.

Introduction

Our aim here is to illustrate a few properties of stationary time series with supporting real life examples. Concepts of auto covariance and auto correlation are shown to be useful which can be easily introduced. In this article we have used number of rainy days and rainfall data of 1979 to 2004 at Aurangabad district to illustrate most of properties theoretically established.

Basic Concepts

Basic definitions and few properties of stationary time series are given in this section.

Definition 2.1: *Probability space:* ***A probability space is a triplet (Ω, C, P) where***

(1). Ω is a set of all possible results of an experiment;

(2). C is class of subsets of Ω (called events) forming a σ- algebra, i.e.

i). $\Omega \in C$,

ii). $A \in C \Rightarrow A^c \in C$,

iii). $\bigcup_{j=1}^{\infty} A_j \in C$, for any sequence $\{A1, A2, \ldots\} \subseteq C$;

(3). P: $C \rightarrow [0, 1]$ is a function which assigns to each event $A \in C$ a number $P(A) \in [0, 1]$, called the probability of A and such that

i). $P(\Omega)=1$,

ii) If $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint events, then $P(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j)$

A Time Series

Let (Ω, C, P) be a probability space let T be an index set. A real valued time series is a real valued function X(t ,ω) defined on T x Ω such that for each fixed $t \in T$, X(t, ω) is a random variable on (Ω, C, P).

The function X(t, ω) is written as X(ω) or X_t and a time series considered as a collection $\{X_t : t \in T\}$, of random variables **[7]**.

Stationary Time Series

The plot of a time series over a time interval [t, t+h] may sometimes closely resemble a plot at another interval [s, s+h]. This implies that there is temporal homogeneity in the behavior of the series, which is called stationary. For example, the number of personal bankruptcies may be stationary in monthly data. This means that the time series between January to March in one year may resemble June to August of another year. A stationary series should have no discernible trends. For precise definition of stationary, some concepts from probability theory, which are developed later, are needed. An imprecise operational definition of stationary time series is as follows: When the mean: E(Xt), the variance: Var(Xt) and all autocavariances of specified lags(say h): Cov(Xt, Xt+h) do not depend on t, the time at which they are measured, we have a stationary time series.

A process whose probability structure does not change with time is called stationary. Broadly speaking a time series is said to be stationary, if there is no systematic change in mean i.e. no trend and there is no systematic change in variance.

Weak Stationary or Covariance Stationary Time Series

A weaker concept of stationary allows the joint distributions to change some what over time, but requires that E(Xt), Var(Xt) do not change. Also, Cov(Xt, Xt+h) is required to be a function of lag length only, not depend on time t at which it is measured. Many stationary series are also called covariance stationary, wide-sense stationary, or second order stationary in the literature. For the multivariate normal joint distribution, weak and strict stationarity are equivalent.

For a random process to be weakly stationary – also called covariance stationary – there are three requirments:

(i) $E(x_t) = \mu < \infty$ for all t, & is not function of t.
(ii) $E(x_t - \mu)^2 < \infty$ for all t, & is not function of t. } … (1)
(iii) $E(x_t - \mu)(x_{t-h} - \mu) = \Upsilon_h < \infty$ for all t & is not function of t.

Definition 2.5: Strict Stationary Time Series: Strict stationary time series is a stronger concept where the properties are unaffected by a change of time origin. It requires that the joint distribution function F

$F[\, x(t_1), x(t_2), \ldots x(t_n)] = F[\, x(t_{1+h}), x(t_{2+h}), \ldots x(t_{n+h})]$ … (2)

for all choices of time points x_1 to x_n and for all h. If the first two moments of strictly stationary process exist, that is they are finite it is also covariance stationary and it satisfies the three conditions (1). Stationary implies stable relations and the object of any theory is to obtain stable relationship among variables.

Main Results

Theorem 3.1: If $\{X_t : t \in T\}$, is strictly stationary with $E\{|X_t|\} < \alpha$ and
$E\{|X_t - \mu|\} < \beta$ then ,
$E(X_t) = E(X_{t+h})$, for all t, h} … (3)
$E[(X_{t1} - \mu)(X_{t2} - \mu)] = E[(X_{t1+h} - \mu)(X_{t2+h} - \mu)]$, for all t_1, t_2, h

Proof follows from definition (2.5).

In usual cases above equation (3) is used to determine that a time series is stationary i.e. there is no trend.

Definition 3.1: Auto-covariance function: The covariance between $\{X_t\}$ and $\{X_{t+h}\}$ separated by h time unit is called auto-covariance at lag h and is denoted by $\Upsilon(h)$.

$\Upsilon(h) = Cov(X_t, X_{t+h}) = E\{X_t - \mu\}\{X_{t+h} - \mu\}$ … (4)

the function $\Upsilon(h)$ is called the auto covariance function.

Theorem 3.2:(Properties of the autocovariance function). Let $\{X_t : t \in T\}$ be a second- order stationary process. The autocovariance function $\Upsilon(h)$ of the process satisfies the following properties:

(1) $\Upsilon(0) = Var(X_t) \geq 0$, for all $t \in T$;
(2) $\Upsilon(h) = \Upsilon(-h)$, for all $h \in Z$ (i.e. $\Upsilon(h)$ is an even function of h);
(3) $\Upsilon(h) \leq \Upsilon(0)$, for all $h \in Z$;
(4) The function $\Upsilon(h)$ is positive semi-definite, i.e.

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_k \Upsilon(t_j - t_k) a_j \geq 0,$$ for any positive integer n and for all the vectors

$a = (a_1, a_2, \ldots a_n)$ and any set of indices $(t_1, t_2 \ldots t_n) \in T$ such that $(t_j - t_k) \in X$.

Definition 3.2: The auto correlation function: The correlation between observation which are separated by h time unit is called auto-correlation at lag h. It is given by

$$P(h) = \frac{E\{X_t - \mu\}\{X_{t+h} - \mu\}}{[E\{X_t - \mu\}^2 E\{X_{t+h} - \mu\}^2]^{1/2}} \quad \ldots (5)$$

$$= \frac{\Upsilon(h)}{[E\{X_t - \mu\}^2 E\{X_{t+h} - \mu\}^2]^{1/2}}$$

where μ is mean .

Remark 3.1: For a stationary time series the variance at time (t + h) is same as that at time t, **[2, 3, 4]**. Thus, the auto correlation at lag h is

$$\rho(h) = \frac{\Upsilon(h)}{\Upsilon(0)} \quad \ldots (6)$$

Remark 3.2: For h = 0, we get, $\rho(0) = 1$.

For application, attempts have been made to establish that stationary satisfy equation (2) and (6).

Theorem 3.3: The covariance of a real valued stationary time series is an even function of h.

i. e. $\Upsilon(h) = \Upsilon(-h)$.

Proof: We assume that without loss of generality, $E\{X_t\} = 0$, then since the series is stationary we get , $E\{X_t X_{t+h}\} = \Upsilon(h)$, for all t and t + h contained in the index set. Therefore if we set $t_0 = t_1 - h$,

$\Upsilon(h) = E\{X_{t0} X_{t0+h}\} = E\{X_{t1} X_{t1-h}\} = \Upsilon(-h)$ … (7)

proved.

Definition 3.3: Positive semi-definite: A function f(x) defined for $x \in X$ is said to be positive semi-definite if it satisfies

$$\sum_{J=i}^{n} \sum_{k=1}^{n} a_k^T f(t_j - t_k) a_j \geq 0,$$

for any set of real vectors $(a_1, a_2, \ldots a_n)$ and any set of indices $(t_1, t_2 \ldots t_n) \in T$ such that $(t_j - t_k) \in X$.

Theorem 3.4: The covariance function of stationary time series $\{X_t : t \in T\}$ is positive semi-definite function in that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_k \Upsilon(t_j - t_k) a_j \geq 0 ,$$

for any set of real $(a_1, a_2, \ldots a_n)$ and any set of indices $(t_1, t_2 \ldots t_n) \in T$.

Proof: The result can be obtained by evaluating the variance of

$$X = \sum_{j=1}^{n} a_j X_{tj.}$$

For this without loss of generality $E(X_t) = 0$. It shows that the variance of a random variable is non-negative i.e. $V(X) \geq 0$.

$$V(X) = V(\Sigma\ a_j X_{tj}) \geq 0$$

$$= E\ (\sum_{j=1}^{n} a_j X_{tj})\ (\sum_{k=1}^{n} a_j X_{tj}) \geq 0,$$

$$= \sum_{J=1}^{n}\sum_{k=1}^{n} a_j a_k\ E\{X_{tj} X_{tk}) \geq 0,$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} a_k\ \Upsilon(t_j - t_k)\ a_j \geq 0 \qquad \dots (8)$$

Hence proved.

Theorem 3.5: $|\rho_{12}(h)| \leq 1$.

Proof: If we set n = 2, in the equation (8) to obtain,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_j \Upsilon(t_i - t_j)\ a_i = a_1^2\, \Upsilon_{11}(0) + a_2^2\, \Upsilon_{22}(0) + 2a_1 a_2\, \Upsilon(t_1 - t_2) \geq 0.$$

$a_1^2\Upsilon_{11}(0) + a_2^2\, \Upsilon_{22}(0) \geq -2a_1 a_2 \Upsilon_{12}(t_1-t_2)$, since $\Upsilon_{11}(0) = \Upsilon_{22}(0)$

$$1/2(a_1^2+a_2^2) \geq \frac{-a_1 a_2 \Upsilon_{12}(t_1-t_2)}{\Upsilon_{11}^{(0)}}$$

Now, let $a_1 = a_2 = 1$ and $t_1 - t_2 = h$,

$$1 \geq \frac{-\Upsilon_{12(h)}}{\Upsilon_{11}^{(0)}} = -\rho_{12}(h) \qquad \dots (9)$$

Similarly, $-a_1 = a_2 = 1$; it shows that

$$P_{12}(h) \leq 1 \qquad \dots (10)$$

From (9) and (10) we get

$$|\rho_{12}(h)| \leq 1.$$

Hence proved.

Theorem 3.6: Let X_t's be independently and identically distributed with $E(X_t) = \mu$ and $var(X_t) = \sigma^2$ then

$$\Upsilon(t, h) = E(X_t, X_h) = \sigma^2, \quad t = h$$
$$= 0, \quad t \neq h$$

This process is stationary in the strict sense.

Testing Procedure

4.1: Inference Concerning Slope (β_1): We set up null hypothesis for testing H_0: $\beta_1 = 0$ Vs H_1: $\beta_1 > 0$ for $\alpha = 0.05$ percent level using t distribution with degrees of freedom is equal to n – 2 were considered. The hypothesis H_0 is not significant for both the values of t for 24 and 14 d. f. for the district.

$$t_{n-2} = \beta_1 / Syx\{1/\Sigma(X - \bar{X})\}^2$$

where β_1 is the slope of the regression line.

4.2: Example of Time Series: Rainfall and number of rainy days data were collected from Vasantrao Naik Marathwada Agricultural University, Parbhani **[1]**. Hence we have two dimensional time series t_i, i = 1, 2 corresponding to the district Aurangabad. Table 5.1A and Table 5.1B shows the results of descriptive statistics, Table 5.1C and Table 5.2C shows linear trend analysis. All the linear trends were found to be not significant.

Over the years many scientists have analyzed rainfall, temperature, humidity, agricultural area, production and productivity of region, **[2, 3, 4, 6]**. Most of them have treated the time series for each of the revenue districts as independent time series and tried to examine the stability or non-stability depending upon series. Most of the times non- stability has been concluded, and hence possibly any sort of different treatment was possibility never thought of. In this investigation we treat the series first and individual series. The method of testing intercept ($\beta_0 = 0$) and regression coefficient ($\beta_1 = 0$), Hooda R.P. **[8]** and for testing correlation coefficient Bhattacharya G.K. and Johanson R.A**. [5].**

The regression analysis tool provided in **MS-Excel** was used to compute β_0, β_1, corresponding SE, t-values for the coefficients in regression models. Results are reported in Table 5.1C and Table5.2C. Elementary statistical analysis is reported in Table5.1A. It is evident from the values of CV that there is hardly any scatter of values around the mean indicating that all the series are not having trend.

Table5.1C shows that the model,

$$X_t = \beta_0 + \beta_1\ t + \in,$$

When applied to the data indicates H_0: $\beta_1 = 0$ is true. Hence X_t is a not having trend for two series of the district. where,

1. X_t are the annual rainfall or number of rainy days series.
2. t is the time (years) variable.
3. is a random error term normally distributed with mean 0 and variance σ^2.

Rainfall or number of rainy days series X_t is the dependent variable and time t in (years) is the independent variable.

Values of auto covariance computed for various values of h are given in Table-5.2A. Rainfall or number of rainy days values for district was input as a matrix to the software. Defining

$A = x_1, x_2 \dots\dots x_{n-h}$

$B = x_{h+1}, x_{h+1} \dots\dots x_n$

$\Upsilon(h) = cov(A, B)$ were computed for various values of h. Since the time series constituted of 26 values, at least 9 values were included in the computation. The relation between $\Upsilon(h)$ were examined using model, Table-5.2C.

$$\Upsilon(h) = \beta_0 + \beta_1\ h + \in,$$

the testing shows that, both the hypothesis $\beta_0 = 0$ and $\beta_1 = 0$ test is not positive. Table-5.2C was obtained by regression values of $\Upsilon(h)$ and h, using "Data Analysis Tools" provided in **MS Excel**. Table 5.2A formed the input for table 5.2C. In other wards, $\Upsilon(h)$ are all zero in the rainfall and number of rainy days series of the district, trends were not found showing that X_t, X_{t+h} are not dependent in both the series of the district and there is no trend in that series.

Conclusion

It was observed that t values are therefore not significant for both the series of the district, i.e. concluded that X_t does not depend on t for both the series of the district [5]. Similarly, $\Upsilon_{ij}(h)$ does not depend on h to mean that , 'no linear relation' rather than 'no relation'. The testing shows that, for the hypothesis $\beta_1 = 0$, test is not positive for t and h for both the series of the district.

Generally it is expected, rainfall or number of rainy days (annual) over a long period at any region to be stationary time series. The results are conform with these series in the district i, e. in the district trends were not found in both the series.

5.1. Rainfall or number of rainy days time series treated as scalar time series

Table 5.1 contains the results for scalar series approach.

The model considered was:

$X_i(t) = (\beta_0)_i + (\beta_1)_i t + \in_i(t), \quad i = 1, 2 \dots (11)$

Where X_i is the annual rainfall or number of rainy days series, t is the time seies variable, β_0 = the intercept, β_1 = the slope, $\in_i$ is the random error. Rainfall X_i is the dependent variable and time t in years is the independent variable.

Table-5.1: Annual rainfall and number of rainy days data of Aurangabad district.

Sr. No.	District→ Years ↓	Number of rainy day	Rainfall
1	1979	47	1127.5
2	1980	39	688
3	1981	51	778.2
4	1882	41	663.2
5	1983	49	905
6	1984	26	370.9
7	1985	19	289.9
8	1986	21	461.4
9	1987	47	851
10	1988	45	823
11	1989	40	818.4
12	1990	45	1102.2
13	1991	33	766.2
14	1992	31	804.1
15	1983	51	901.6
16	1994	46	691.4
17	1995	33	636.4
18	1996	46	735
19	1997	49	733.9
20	1998	57	932.9
21	1999	46	695
22	2000	38	693
23	2001	42	651.2
24	2002	37	688.7
25	2003	40	634
26	2004	41	694

Decriptive Statistics:

Table-5.1A: Elementary statistics of observed minimum, maximum, mean, standard deviation (S.D.) and coefficient of variation (C.V.)of number of rainy days of Aurangabad district.

Aurangabad District	Minimum	Maximum	Mean	S.D.	C.V.
Rainydays:	19	57	40.8	9.1	22.2

Table-5.1B: Elementary statistics of observed minimum, maximum, mean, standard deviation (S.D.) and coefficient of variation (C.V.)of annual rainfall of Aurangabad district.

Aurangabad	Minimum (mm)	Maximum (mm)	Mean (mm)	S.D.(mm)	C.V.
Rainfall	289.9	1127.5	736.0	182.9	24.8

Table-5.1C: Linear regression analysis of data to determine trend Eq(11)

Aurangabad District	Coefficients		Standard Error	t Stat	Significance	P Value	Lower 95%	Upper 95%
Rainydays	β_0	38.6	3.8	10.2*	S	0.0	30.8	46.4
	β_1	0.2	0.2	0.7	NS	0.5	-0.3	0.7
Rainfall	β_0	759.6	76.7	9.9*	S	0.0	601.4	917.9
	β_1	-1.8	5.0	-0.4	NS	0.7	-12.0	8.5

*t =2.04 is the critical value for 24 d f at 5% L. S. * shows the significant value*

A look at the table 5.1A &B shows that all of them have similar values of CV. Which indicates that their dispersion is almost identical. Trends were found to be not significant in both the series of the district. In absence of linear trend, with reasonably low CV values can be taken as evidence of series being stationary series individually in the district.

Further search for evidences of stability included determination of auto covariance and their dependency on lag variable h (Table 5.2A). Such an analysis requires an assumption of AR(Auto-regressive) model **[7]** Eq(12). Therefore a real test for stationary property of the time series can come by way of establishing auto- covariance's which do not depend on the lag variable

$X_t = C + \Phi X_{t-h} + € _t, \ h = 0, 1, 2, \dots .15 \dots \quad (12)$

Table-5.2A : Auto variances: Individual column treated as ordinary time series for lag values (h = 0, 1, 2, … 15) about both the data.

Nanded District		
lag h	Number of rainy days	Rainfall
0	81.9	33438.3
1	26.4	9419.9
2	-5.5	2813.2
3	-19.2	-3898.4
4	-10.6	-2971.8
5	-6.5	-15178.1
6	7.8	-14864.5
7	-3.2	-8058.1
8	-8.6	-821.1
9	-2.0	1859.2
10	8.4	4391.0
11	2.3	8341.1

12	-36.3	-6359.4
13	-40.0	-8175.9
14	-16.9	-1365.9
15	20.3	4668.3

Table-5.2B : Correlation coefficient between h and Auto covariance is :

Corr. Coefficient	-0.47*	-0.29

*Correlation coefficient r = 0.433 is the critical value for 14 d f at 5% L. S. * shows the significant value.*

Correlation's between Υ_{ij} **(h)** and **h** were found to be ***significant*** in number of rainy days but found to be not ***significant*** the rainfall series of the district. It is seen that the rainfall time series can be reasonably assumed to be **stationary** i.e. not having trend**.**

Table-5.2C : Linear regression analysis of lag values h vs covariance.

Aurangabad District	Coefficients		Standard Error	t Stat	Significance	P Value	Lower 95%	Upper 95%
Rainydays	β_0	20.7	12.2	1.7	NS	0.1	-5.5	47.0
	β_1	-2.8	1.4	-2.0	NS	0.1	-5.8	0.2
Rainfall	β_0	5500.3	5426.6	1.0	NS	0.3	-6138.6	17139.3
	β_1	-706.4	616.4	-1.1	NS	0.3	-2028.5	615.7

t =2.1 is the critical value for 14 d f at 5% L. S.

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